

More Transmission Line Math Than You Want

Sam Wetterlin

11/26/09

(This document is a not-especially-well organized collection of useful equations for transmission lines.)

Transmission lines commonly take the form of coax cable, twisted pairs, or parallel conductors. They have a characteristic impedance that determines the voltage-current relationship of the signal propagating down the line. For example, in a 50-ohm coax cable, the propagating signal voltage is V volts, then the current is $V/50$ amps.

The characteristic impedance is often just called impedance. Technically, however, the impedance of a transmission line, as for any component, is the extent to which it impedes the signal transfer. The impedance of a transmission line is hopefully near zero. In a transmission line with loss, the longer the transmission line, the greater the loss and the greater the impedance. The characteristic impedance, however, is independent of length, and is a characteristic of that particular type of line.

Characteristic impedance is often referred to as Z_0 , but it is typically assumed to be a pure resistance and can also be referred to as R_0 . Short transmission lines are typically assumed to be lossless. If a short line has significant loss, then its Z_0 will contain a significant reactive component. So lossy transmission lines have more complications than just the loss they introduce. We will see below that Z_0 of a normal coax can have a very significant reactive component below 1 MHz, and even as high as 10 MHz.

The major significance of Z_0 is that a transmission line terminated in its Z_0 will produce no reflections at the termination. Another way to state this is that a 50-ohm line terminated with 50 ohms will present an impedance of 50 ohms at its input. Any other terminating impedance may be “transformed” by the transmission line. Thus, the input impedance of a 50-ohm line terminated with 25 ohms will generally be neither 25 nor 50 ohms; its value will depend on the length of the line. A lossless line that is $1/4$ wavelength long makes an especially nice transformer, causing a terminating resistance of R to look like $2500/R$ ohms. That is,

$$Z_{in} = Z_0^2 / Z_L \quad (\text{Equation 1})$$

Z_{in} is the input impedance, and Z_L is the terminating impedance. A useful way to look at this is that if the terminating impedance is Z_0/N , the input impedance will be Z_0*N . If you start out at half of Z_0 , you end up at double Z_0 . Small impedances look big, and big impedances look small.

Equation 1 applies to lossless coax that is one-quarter wavelength long. To deal with real coax cable, it is sometimes necessary for the MSA to utilize an equation that takes into account the loss factor. Coax loss is usually measured in dB per hundred feet, a parameter called A_0 . To actually use the loss factor in an equation, though, it is converted to dB per foot and then to nepers/foot (a neper is $20/\ln(10)=8.686$ dB), and labeled α . The conversion equation is

$$\alpha = 0.0011513 * A_0 \quad (\text{Eq A-1; loss factor in nepers/ft})$$

A second factor labeled β that is also used to characterize the transmission line. It describes the number of radians of phase delay are caused by one foot of the transmission line. It depends on

the velocity factor (VF), which is the proportion of the speed of light at which signals propagate on the transmission line:

$$\beta = \frac{2\pi f}{VF * 983.6} \quad (\text{Eq A-2; radians/ft; } f \text{ is frequency in MHz})$$

The equation for β is based on the fact that light travels 983.6 million feet per second.

The two factors α and β are put together into a single “propagation constant” called γ (gamma):

$$\gamma = \alpha + j\beta \quad (\text{Eq A-3; propagation constant})$$

The propagation constant combines the loss per foot with the number of radians per foot (at a given frequency). Therefore, if the transmission line is L feet long, γL will provide the total loss incurred and the total phase delay in a one-way trip down the transmission line. The quantity γL is the one that usually appears in the transmission line equations.

Unsimplified--General Equation for Input Impedance

Now that we have defined the propagation constant, we can state a general, and ugly, equation for the input impedance Z_{in} of a transmission line with characteristic impedance Z_0 terminated by specified impedance, Z_L :

$$Z_{in} = Z_0 \cdot \frac{Z_L \cosh(\gamma L) + Z_0 \sinh(\gamma L)}{Z_L \sinh(\gamma L) + Z_0 \cosh(\gamma L)} \quad (\text{Eq A-4; } Z_{in} \text{ of any terminated transmission line})$$

Not only are the hyperbolic sine and cosine functions a bit exotic, but here they have complex arguments, and therefore produce complex results. Likewise, all the Z's can be complex numbers. That is why the first thing we usually do is make some simplifying assumptions.

Simplified— Z_{in} of Lossless Line

For Equation 1 in the main text, we assumed that the transmission line is lossless and is $\frac{1}{4}$ wavelength long, which produced a much simpler equation. We can assume the transmission line is lossless and of arbitrary length L (in feet) and get a slightly more complicated equation:

$$Z_{in} = Z_0 \cdot \frac{Z_L + jZ_0 \tan(\beta L)}{Z_0 + jZ_L \tan(\beta L)} \quad (\text{Eq A-5; } Z_{in} \text{ of lossless line of any length})$$

Not only is the form of this equation simpler than Eq A-4, but we are now dealing with a plain-old tangent of a real number. Remember that β is the phase rate in radians per foot, so βL is the phase delay of the line. The significance of the common assumption that a line is $\frac{1}{4}$ wavelength long is that βL becomes $\pi/2$ radians, simplifying the trigonometry.

Simplified— Z_{in} With Only Opens and Shorts Allowed

Alternatively, we can allow the line to have loss, but restrict it to being terminated by either an open circuit or a short circuit:

$$Z_{open} = Z_0 / \tanh(\gamma L) \quad (\text{Eq A-6; input } Z \text{ of lossy line of any length, open circuited})$$

$$Z_{\text{short}} = Z_0 \cdot \tanh(\gamma L) \quad (\text{Eq A-7; input } Z \text{ of lossy line of any length, short circuited})$$

We are now back to hyperbolic tangents, but at least the argument is real.

Unsimplified—The Inverse Equations

It is sometimes handy to solve for something other than Z_{in} in Eq A-4. First, we might want to solve for the terminating impedance required to produce a given input impedance. We can algebraically manipulate Eq A-4 into this form, but it is actually simpler to use a transmission line of negative length:

$$Z_L = Z_0 \cdot \frac{Z_{\text{in}} \cosh(-\gamma L) + Z_0 \sinh(-\gamma L)}{Z_{\text{in}} \sinh(-\gamma L) + Z_0 \cosh(-\gamma L)} = Z_0 \cdot \frac{Z_{\text{in}} \cosh(\gamma L) - Z_0 \sinh(\gamma L)}{Z_0 \cosh(\gamma L) - Z_{\text{in}} \sinh(\gamma L)} \quad (\text{Eq A-8})$$

We also might want to solve for Z_0 , given all the other values:

$$Z_0 = \frac{(Z_{\text{in}} - Z_L) \coth(\gamma L) \pm \sqrt{(Z_L - Z_{\text{in}})^2 \coth^2(\gamma L) + 4Z_{\text{in}}Z_L}}{2} \quad (\text{Eq A-9})$$

Note that this last equation has two solutions. The trick is to pick the one with a positive real part. Using the “+” sign usually will do it. If there is any way for the solution to involve using the negative radical, it would probably require extremely odd and complex value for Z_L .

Simplified—Finding Z_0 of $\frac{1}{4}$ Wavelength Line

We can simplify Eq A-9 by assuming the line is $\frac{1}{4}$ wavelength long, which produces the following:

$$Z_0 = \frac{(Z_{\text{in}} - Z_L) \tanh(\alpha L) + \sqrt{(Z_L - Z_{\text{in}})^2 \tanh^2(\alpha L) + 4Z_{\text{in}}Z_L}}{2} \quad (\text{Eq A-10; } Z_0 \text{ of } \frac{1}{4}\text{-wave line})$$

Note the tanh has a real argument and a real result. If we use a resistance for Z_L , only Z_{in} (and Z_0) may be complex numbers. In the real world of measurement errors, this equation works best if the terminating impedance, Z_L , is a resistance within a factor of 2:1 of the value of Z_0 , although using an open circuit as the termination and calling it 10^7 ohms often works reasonably well.

Calculating Complex Z_0 from Basic Parameters R, G, L and C

In the above equations, we were calculating various impedances from other impedances. It is also possible to calculate Z_0 from four characteristics of the transmission line:

- L=inductance in uH per foot
- C=capacitance in pF per foot
- R=resistance in ohms/foot
- G=conductance in siemens/foot

L and C are relatively fixed, but R and G vary with frequency. R is composed of the DC resistance (usually considered to be zero) and the skin-effect resistance, which is proportional to

the square-root of frequency (since the skin depth itself is so proportional). G is the conductance of the dielectric, which is proportional to frequency. It is a little difficult to view dielectric losses as being current flow through the dielectric. Just think of G as being used to represent the fact that dielectric losses are proportional to voltage, as would the current flow through a resistance in the dielectric.

With these four factors, we can calculate the characteristic impedance from the frequency (in MHz)

$$Z_0 = \sqrt{\frac{R + j2\pi fL}{G + j2\pi fC \cdot 10^{-6}}} \quad (\text{Eq A-11; } Z_0 \text{ from basic parameters})$$

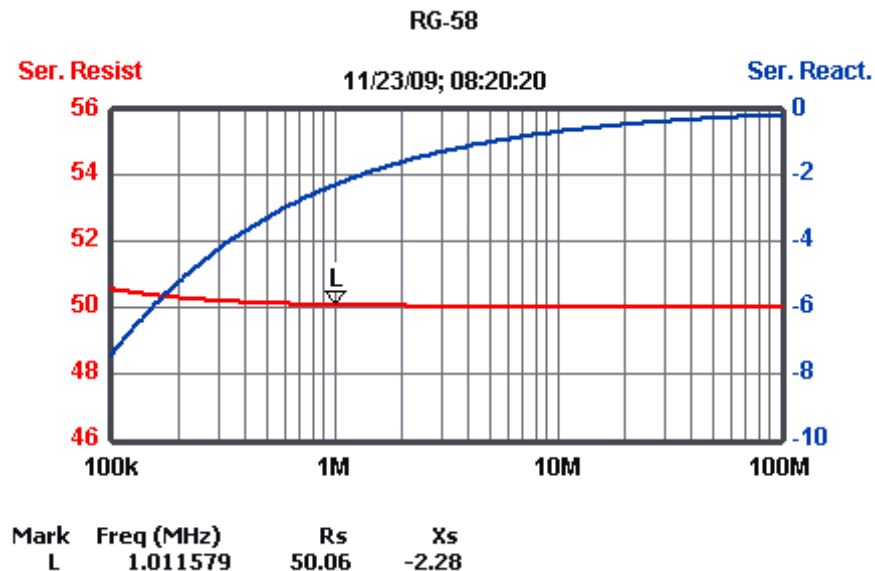
(R is ohms/foot; G is siemens/foot; L is uH per foot and C is pF per foot. f is in MHz.)

Note that this equation produces a complex number unless R and G are zero. One of the benefits of assuming a line is lossless is to make them zero and get a real value for Z_0 . In the real world, there are losses, but at high enough frequency $2\pi Lf$ becomes big enough to swamp R, and $2\pi Cf$ becomes big enough to swamp G, producing a characteristic impedance that approaches

$$Z_0 = \sqrt{\frac{L}{C \cdot 10^{-6}}} \quad (\text{Eq A-12; high frequency } Z_0)$$

L is here in uH/ft and C in pF/ft; if H/ft and F/ft are used instead, the 10^{-6} factor disappears.

Here is the simulated Z_0 for RG-58 coax (there are many flavors of RG-58; this is just one of them):



By 1 MHz, the real part of Z_0 has settled near its nominal value of 50 ohms, but there is still a significant imaginary component. By about 20 MHz that imaginary component is less than $\frac{1}{2}$ ohm. It is rare that we actually do computations with the imaginary component—we normally assume it to be zero. The above simulation shows that at 1 MHz a coax cable terminated with 50

ohms might produce more reflections than you would have expected by treating it as true 50-ohm cable.

Calculating Complex Z_0 from Loss Factors

Some of the calculations done within the MSA do take into account the imaginary part of Z_0 . This leads to the question as to how to calculate that component, given that we may be able to find values for L and C in A-11, but it is not so easy to find values for R and G. It's simple: we pull ourselves up by our bootstraps and calculate R, G, L and C from Z_0 , and then calculate Z_0 from R, G, L and C.

To do this we use these equations:

$$L = \frac{\text{Im}(\gamma \cdot Z_0)}{2\pi f}$$

$$C = \frac{\text{Im}(\gamma / Z_0)}{2\pi f \cdot 10^{-6}}$$

$$R = 2\alpha_c \cdot \text{Re}(Z_0)$$

$$G = \frac{2\alpha_D \cdot \text{Re}(Z_0)}{|Z_0|^2}$$

γ is our old friend, the propagation constant. The real part of γ , which we have called α , can be broken into two components, one representing the loss (nepers/ft) caused by the conductor losses of the transmission line, and the other caused by the dielectric losses; these are called α_c and α_D , respectively. These in turn are calculated from loss factors called K_1 and K_2 , which are available from a variety of sources:

$$A_0 = A_C + A_D \quad \text{loss in dB per hundred feet}$$

$$A_C = K_1 \cdot \sqrt{f} \quad f \text{ in MHz}$$

$$A_D = K_2 \cdot f \quad f \text{ in MHz}$$

$$\alpha = \alpha_c + \alpha_D \quad \text{loss in nepers per foot}$$

$$\alpha_c = 0.0011513 \cdot A_C$$

$$\alpha_D = 0.0011513 \cdot A_D$$

The K_1 and K_2 values are available for many types of coax cable. Here they are used in a formula to calculate loss per hundred feet based on frequency in MHz. In some programs, the formula calculates loss per foot based on frequency in GHz, which results in different K factor values.

We proceed by iteration, starting with Z_0 equal to its nominal real value. We then calculate R, G, L and C, and then use those to calculate a second value for Z_0 . This second value is often remarkably accurate, but we repeat the process to get a third value for Z_0 . For good measure, repeat the process one more time for large loss factors or frequencies below 5 MHz.

Q and Loss Factor

Q of open and short stubs can be calculated as follows:

$$Q = \alpha / (2\beta)$$

$$Q = 2.774 * F_q / (A_0 * VF) \quad F_q = \text{quarter wave freq (MHz)}; VF = \text{velocity factor}$$

$$\text{But, } VF = \frac{L \cdot F_q}{245.9}, \text{ so}$$

$$Q = 682.2 / (A_0 * L) \quad L = \text{length in feet of a quarter-wave stub}$$

Alternatively, we can determine Q by measurements, and use the above formula to calculate A_0 .

Q in the area of a quarter-wave stub (open or short) is the same as that for a series LC circuit, or

$$Q = \frac{|X| + \omega \frac{dX}{d\omega}}{2R_s} \quad (X \text{ is reactance, } \omega \text{ is freq (radians/sec), } R_s \text{ is series resistance)}$$

Q in the area of a half-wave stub (open or short) is the same as that for a parallel LC circuit, or

$$Q = \frac{|S| + \omega \frac{dS}{d\omega}}{2G} \quad (S \text{ is susceptance, } \omega \text{ is freq (radians/sec), } G \text{ is conductance)}$$

Once we have measured Q, from above we have

$$A_0 = 682.2 / (Q * L) \quad L \text{ is length in feet}$$

S21

In a system with reference impedance Z_0 , S_{21} of a coax with characteristic impedance Z_C is:

$$S_{21} = \frac{2}{e^{\gamma L} + e^{-\gamma L} + \frac{e^{\gamma L} - e^{-\gamma L}}{2} \left(\frac{Z_C}{Z_0} + \frac{Z_0}{Z_C} \right)}$$

$$S_{21} = \frac{2}{2 * \cosh(\lambda L) + \sinh(\lambda L) \left(\frac{Z_c}{Z_0} + \frac{Z_0}{Z_c} \right)}$$

See **Estimation of Printed Circuit Board Power Bus Noise at Resonance Using a Simple Transmission Line Model**, www.evel.clemson.edu/pdf/EMCS01-896.pdf